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Spherically Symmetric Conformally Flat Distributions of Charged Dust and Zero-Mass Scalar Fields

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The problem of charged dust distribution in the presence of zero-mass scalar field for a spherically symmetric conformally flat metric has been investigated. Exact solutions are obtained in the comoving coordinate system for the static model as well as for the nonstatic model. It has been shown that in the nonstatic model the electromagnetic field and dust distribution cannot survive when the scalar field is taken to be a function of time t only. Physical interpretation of the solutions has been investigated.

1. INTRODUCTION

In the present paper, we have investigated the problem of zero-mass scalar field interactions in the presence of charged dust distribution. Exact solutions are obtained in the comoving coordinate system. We have studied the problem in the static model as well as in the nonstatic model. In the nonstatic model if the scalar field is taken to be a function of time t only, it is shown that both the electromagnetic field and the dust distribution cannot survive.

In Section 2 the formulation of the problem is presented. In Section 3 the field equations are given. Solutions and their physical interpretation are presented in Section 4.

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2. FORMULATION OF THE PROBLEM

Einstein's field equations are given by

$$G_{ij} \equiv R_{ij} - (1/2)Rg_{ij} = -k[T_{ij} + T'_{ij} + T''_{ij}] \tag{1}$$

where the energy–momentum tensors of zero-mass scalar field, the electromagnetic field, and dust distribution are given by

$$T_{ij} = \frac{1}{4\pi}\left(V_{,i}V_{,j} - g_{ij}V_{,k}V^{,k}\right) \tag{2}$$

$$T'_{ij} = \frac{1}{4\pi}\left(-F_{i\alpha}F_j^{\alpha} + \tfrac{1}{4}g_{ij}F_{\alpha\beta}F^{\alpha\beta}\right) \tag{3}$$

and

$$T''_{ij} = \rho U_i U_j \tag{4}$$

respectively.

Here ρ is the mass density and U_i is the 4-velocity vector.

The scalar V satisfies the wave equation

$$g^{ij}V_{;ij} = 0 \tag{5}$$

The electromagnetic field equations are given by

$$F^{ij}_{;j} = -\sigma' U^i \tag{6}$$

and

$$F_{ij,k} = 0 \tag{7}$$

where σ' is the charge density.

The line element considered for the problem is

$$ds^2 = e^{\lambda}\left[dt^2 - dr^2 - r^2\, d\theta^2 - r^2 \sin^2\theta\, d\varphi^2\right] \tag{8}$$

where λ is a function of r and t only.

As for notation a comma or a semicolon followed by a subscript denotes partial differentiation or covariant differentiation, respectively. Units are so chosen that $C = G = 1$.

3. FIELD EQUATIONS

Considering the comoving coordinate system we get

$$U_1 = U_2 = U_3 = 0 \qquad \text{and} \qquad U_4 = e^{\lambda/2} \tag{9}$$

For the metric (8) the surviving field equations are given by

$$G_{11} \equiv \frac{3}{4}\lambda^2_{,1} + \frac{2\lambda_{,1}}{r} - \frac{\lambda^2_{,1}}{4} - \lambda_{,44} = k\left[\frac{1}{8\pi}\left(V_{,1}^2 + V_{,4}^2\right) + T'_{11}\right] \tag{10}$$

$$G_{22} \equiv \lambda_{,11} - \lambda_{,44} + \frac{\lambda_{,1}}{r} - \frac{\lambda^2_{,4}}{4} + \frac{\lambda^2_{,1}}{4} = k\left[\frac{1}{8\pi}\left(-V_{,1}^2 + V_{,4}^2\right) + T'_{22}\right] \tag{11}$$

$$G_{33} \equiv G_{22} \tag{12}$$

$$G_{44} \equiv -\lambda_{,11} + \frac{3}{4}\lambda^2_{,4} - \frac{2\lambda_{,1}}{r} - \frac{\lambda^2_{,1}}{4} = k\left[\frac{1}{8\pi}\left(V_{,1}^2 + V_{,4}^2\right) + \rho e^{\lambda} + T'_{44}\right] \tag{13}$$

$$G_{14} \equiv \lambda_{,14} - \frac{1}{2}\lambda_{,1}\lambda_{,4} = -k\left(\frac{1}{4\pi} V_{,1} V_{,4} + T'_{44}\right) \tag{14}$$

$$G_{12} \equiv 0 = F_{14}F_{24} - \frac{F_{13}F_{32}}{r^2 \sin^2\theta} \tag{15}$$

$$G_{13} \equiv 0 = F_{14}F_{34} - \frac{1}{r^2} F_{12}F_{32} \tag{16}$$

$$G_{23} \equiv 0 = F_{24}F_{34} - F_{21}F_{31} \tag{17}$$

$$G_{24} \equiv 0 = F_{21}F_{41} + \frac{1}{r^2 \sin^2\theta} F_{23}F_{43} \tag{18}$$

and

$$G_{34} \equiv 0 = F_{31}F_{41} + \frac{1}{r^2} F_{32}F_{42} \tag{19}$$

where

$$V_{,2} = 0, \qquad V_{,3} = 0, \qquad V_{,1} = \frac{\partial V}{\partial r}, \qquad \text{and} \qquad V_{,4} = \frac{\partial V}{\partial t}$$

From (6) we obtain the equations

$$F^{41}_{,1}+F^{41}\left(2\lambda_{,1}+\frac{2}{r}\right)=-\sigma' U^4 \tag{20}$$

$$\tfrac{3}{2}F^{14}\lambda_{,4}+F^{14}_{,4}=0 \tag{21}$$

and

$$F^{32}_{,2}+F^{32}\cot\theta=0 \tag{22}$$

From (7), we obtain

$$\frac{\partial F_{14}}{\partial\theta}=\frac{\partial F_{14}}{\partial\varphi}=0 \tag{23}$$

and

$$\frac{\partial F_{23}}{\partial t}=\frac{\partial F_{23}}{\partial r}=0 \tag{24}$$

The equation (5) becomes

$$V_{,44}-V_{,11}-\frac{2V_{,1}}{r}-\lambda_{,1}V_{,1}+V_{,4}\lambda_{,4}=0 \tag{25}$$

4. SOLUTIONS OF THE FIELD EQUATIONS AND THE PHYSICAL INTERPRETATIONS OF THE SOLUTIONS

From equations (15)–(19), we obtain

$$F_{12}=F_{13}=F_{24}=F_{34}=0,\qquad F_{14}\neq 0 \text{ and } F_{23}\neq 0 \tag{26}$$

Using (26) in (3), we obtain

$$-T'_{11}=T'_{22}=T'_{33}=T'_{44}=\frac{e^{-\lambda}}{8\pi}\left(F_{14}^2+\frac{F_{23}^2}{r^4\sin^2\theta}\right)$$

$$T'_{rs}=0,\qquad (r\neq s;\ r,s=1,2,3,4) \tag{27}$$

From (21) and (23), we obtain

$$F_{14}=e^{\lambda/2}f(r) \tag{28}$$

where $f(r)$ is an arbitrary function of r.

From (24) and (22), we obtain

$$F_{23}=L\sin\theta \tag{29}$$

where L is an arbitrary constant.

To solve the field equations completely we consider the cases where λ is a function of r only or a function of time t only.

Using (27) and taking λ as a function of r or t only in (14), we obtain

$$V_{,1}V_{,4}=0 \tag{30}$$

which shows that either $V_{,1}=0$ or $V_{,4}=0$.

Case I. $\lambda=\lambda(r)$, $V_{,1}\neq 0$, $V_{,4}=0$. Using the above conditions and (29) in (10), (11), and (13), we obtain

$$\frac{3}{4}\lambda_{,1}^2+\frac{2\lambda_{,1}}{r}=\frac{k}{8\pi}\left[V_{,1}^2-e^{-\lambda}\left(F_{14}^2+\frac{L^2}{r^4}\right)\right] \tag{31}$$

$$\lambda_{,11}+\frac{\lambda_{,1}}{r}+\frac{\lambda_{,1}^2}{4}=\frac{k}{8\pi}\left[-V_{,1}^2+e^{-\lambda}\left(F_{14}^2+\frac{L^2}{r^4}\right)\right] \tag{32}$$

and

$$-\lambda_{,11}-\frac{2\lambda_{,1}}{r}-\frac{\lambda_{,1}^2}{4}=\frac{k}{8\pi}\left[V_{,1}^2+e^{-\lambda}\left(F_{14}^2+\frac{L^2}{r^4}\right)\right]+ke^{\lambda}\rho \tag{33}$$

respectively.

From (31) and (32), we obtain the relation

$$\lambda_{,1}^2+\frac{3\lambda_{,1}}{r}+\lambda_{,11}=0 \tag{34}$$

which on solving yields

$$\lambda=\log\left(\beta_1-\frac{\alpha_1}{2r^2}\right) \tag{35}$$

where β_1 and α_1 are arbitrary constants.

The equation (25) in this case reduces to the form

$$V_{,11}+\frac{2V_{,1}}{r}+V_{,1}\lambda_{,1}=0 \tag{36}$$

Substituting the value of λ from (35) in (36) and solving the resulting equation, we obtain

$$V=\frac{A}{(2\alpha_1\beta_1)^{1/2}}\log\left[\frac{r-(\alpha_1/2\beta_1)^{1/2}}{r+(\alpha_1/2\beta_1)^{1/2}}\right]+B \tag{37}$$

where A and B are arbitrary constants.

Using (35), (29), and (37) in (31) or (32), we obtain

$$F_{14}^2=\frac{A^2}{r^4(\beta_1-\alpha_1/2r^2)}-\frac{6\alpha_1^2\pi}{kr^6(\beta_1-\alpha_1/2r^2)}-\frac{16\pi\alpha_1}{kr^4}-\frac{L^2}{r^4} \tag{38}$$

Using (35), (37), and (38) in (33), we obtain

$$k\rho=\frac{3\alpha_1 r^2\beta_1}{r^6(\beta-\alpha_1/2r^2)^3}-\frac{kA^2}{4\pi r^4(\beta_1-\alpha_1/2r^2)^3} \tag{39}$$

Using (38) and (35) in (20), we obtain the charge density as

$$\sigma'=\pm\left(\beta_1-\frac{\alpha_1}{2r^2}\right)^{-3/2}$$

$$\frac{\times\left[\frac{-A^2\alpha_1}{2r^7(\beta_1-\alpha_1/2r^2)^2}+\frac{6\alpha_1^2\pi}{kr^7(\beta_1-\alpha_1/2r^2)}+\frac{3\alpha^3\pi}{kr^9(\beta_1-\alpha_1/2r^2)^2}\right]}{\frac{A^2}{r^4(\beta_1-\alpha_1/2r^2)}-\frac{6\alpha_1^2\pi}{kr^6(\beta_1-\alpha_1/2r^2)}-16\pi\alpha_1/kr^4-L^2/r^4} \tag{40}$$

where the positive sign corresponds to the negative value of F_{14} and the negative sign corresponds to the positive value of F_{14}.

The reality condition obtained from (39) is

$$3\alpha_1\beta_1>\frac{kA^2}{4\pi} \tag{41}$$

From (38) we see that F_{14} will be real provided

$$r^2 > \frac{2\alpha_1^2\pi/k + \alpha_1 L^2/2}{16\pi\alpha_1\beta_1/k + L^2\beta_1 - A^2} \tag{42}$$

The solutions (29), (35), (37), (38), (39), and (40) constitute the complete set of solutions under Case I. We observe from (38), (39), and (40) that F_{14}, ρ, and σ' all decrease with the increase of r and tend to zero as $r \to \infty$.

From (37) we also observe that the scalar V decreases with the increase of r but it reduces to an arbitrary constant as $r \to \infty$.

Case II. $\boldsymbol{\lambda = \lambda(t)}$, $\boldsymbol{V_{,1} = 0}$, $\boldsymbol{V_{,4} \neq 0}$. Using the above conditions the field equations (10), (11), and (13) reduce to

$$-\lambda_{,44} - \frac{\lambda_{,4}^2}{4} = k\left[\frac{1}{8\pi} V_{,4}^2 - \frac{e^{-\lambda}}{8\pi}\left(F_{14}^2 + \frac{L^2}{r^4}\right)\right] \tag{43}$$

$$-\lambda_{,44} - \frac{\lambda_{,4}^2}{4} = \frac{k}{8\pi}\left[V_{,4}^2 + e^{-\lambda}\left(F_{14}^2 + \frac{L^2}{r^4}\right)\right] \tag{44}$$

and

$$\frac{3}{4}\lambda_{,4}^2 = k\left[\rho e^{\lambda} + \frac{1}{8\pi} V_{,4}^2 + \frac{e^{\lambda}}{8\pi}\left(F_{14}^2 + \frac{L^2}{r^4}\right)\right] \tag{45}$$

respectively.

From (43) and (44), we obtain

$$F_{14}^2 + \frac{L^2}{r^4} = 0 \tag{46}$$

which implies that

$$F_{14} = 0 \qquad \text{and} \qquad L = 0 \tag{47}$$

When $L = 0$, we have from (29)

$$F_{23} = 0 \tag{48}$$

The results (47) and (48) show that the electromagnetic field cannot survive for a conformally flat spherically symmetric metric provided the scalar V and λ both are functions of time t only.

The wave equation (25) reduces to the form

$$V_{,44}+V_{,4}\lambda_{,4}=0 \tag{49}$$

which on solving yields

$$V=k_1\int e^{-\lambda}\,dt+k_2 \tag{50}$$

where k_1 and k_2 are arbitrary constants.

Using (50) in (43) or (44), we obtain

$$-\lambda_{,44}-\frac{\lambda_{,4}^2}{4}=\frac{k}{8\pi}k_1^2e^{-2\lambda} \tag{51}$$

A particular solution of (51) is obtained as

$$\lambda=\log(k_3t+k_4) \tag{52}$$

where k_3 and k_4 are arbitrary constants subject to the condition that

$$3k_3^2=\frac{kk_1^2}{2\pi} \tag{53}$$

which is a relation between the constants k, k_1, and k_3.

Using (52) in (50), we obtain

$$V=\frac{k_1}{k_3}\log(k_3t+k_4)+k_2 \tag{54}$$

Using (52), (53), and (54) in (45), we obtain

$$\rho=0 \tag{55}$$

Hence the dust distribution cannot survive as well as electromagnetic field provided the scalar V is taken to be a function of time t only in a purely nonstatic model.

Case III. $\boldsymbol{\lambda=\lambda(t)}$, $\boldsymbol{V_{,4}=0}$, $\boldsymbol{V_{,1}\neq 0}$. The equations (10), (11), and (13) now reduce to

$$-\lambda_{,44}-\frac{\lambda_{,4}^2}{4}=\frac{k}{8\pi}\left[V_{,1}^2-e^{-\lambda}\left(F_{14}^2+\frac{L^2}{r^4}\right)\right] \tag{56}$$

$$-\lambda_{,44}-\frac{\lambda_{,4}^2}{4}=\frac{k}{8\pi}\left[V_{,1}^2+e^{-\lambda}\left(F_{14}^2+\frac{L^2}{r^4}\right)\right] \tag{57}$$

and

$$\frac{3}{4}\lambda_{,4}^{2}=k\left[\rho e^{\lambda}+\frac{1}{8\pi}V_{,1}^{2}+e^{-\lambda}\left(F_{14}^{2}+\frac{L^{2}}{r^{4}}\right)\right] \tag{58}$$

respectively.

From (56) and (57), we obtain

$$V_{,1}^{2}=e^{-\lambda}\left(F_{14}^{2}+\frac{L^{2}}{r^{4}}\right) \tag{59}$$

The equation (25) reduces to the form

$$V_{,11}+\frac{2V_{,1}}{r}=0 \tag{60}$$

which on solving yields

$$V=\frac{k^{5}}{r}+k_{6} \tag{61}$$

where k_5 and k_6 are arbitrary constants.

Using (28) in (59), we obtain

$$V_{,1}^{2}=f^{2}(r)+\frac{e^{-\lambda}L^{2}}{r^{4}} \tag{62}$$

Since V is taken to be a function of r only the relation (62) will be valid only when the arbitrary constant $L=0$.

Taking $L=0$ in (29) and (62), we obtain

$$F_{23}=0 \tag{63}$$

and

$$V_{,1}^{2}=f^{2}(r) \tag{64}$$

But from (61), we obtain

$$V_{,1}=-\frac{k_{5}}{r^{2}} \tag{65}$$

From (64) and (65), we can derive

$$f^2(r)=\frac{k_5^2}{r^4} \tag{66}$$

Using (66) in (68), we obtain

$$F_{14}=\pm\frac{(e^{\lambda/2})k_5}{r^2} \tag{67}$$

Using (59) in (56) or (57), we obtain

$$\lambda_{,44}+\frac{\lambda_{,4}^2}{4}=0 \tag{68}$$

which on solving yields

$$\lambda=4\log(k_7t+k_8) \tag{69}$$

where k_7 and k_8 are arbitrary constants.

Using (69) in (67), we obtain

$$F_{14}=\pm\frac{k_5}{r^2}(k_7t+k_8)^2 \tag{70}$$

Using (69) and (70) in (58), we obtain

$$k\rho=\frac{1}{(k_7t+k_8)^2}\left[\frac{12k_7^2}{(k_7t+k_8)^2}-\frac{kk_5^2}{4\pi r^4}\right] \tag{71}$$

The reality condition obtained from (71) is

$$12k_7^2>\frac{kk_5^2}{4\pi r^4}(k_7t+k_8)^2 \tag{72}$$

Using (69) and (70) in (20) we obtain the charge density σ' to be zero.

Since the charge density is zero we see that the nature of the material is neutral and charge does not reside on the matter. From (61), (70), and (71), we observe that the scalar V and the electric field component F_{14} decrease with the increase of r and the mass density ρ increases with the increase of r. The electric component F_{14} increases with time while the mass density ρ

decreases with time. When $r \to \infty$, we obtain from (61), (70), and (71)

$$V = k_6$$
$$F_{14} = 0 \tag{73}$$
$$\rho = \frac{12k_7^2}{k} \cdot \frac{1}{(k_7 t + k_8)^4}$$

respectively.

The solution (73) shows that an uncharged dust distribution interacting with a constant scalar field V exists as $r \to \infty$.